Realizability of stationary spherically symmetric transonic accretion

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(Received 2 November 2001; revised manuscript received 10 September 2002; published 10 December 2002)

The spherically symmetric stationary transonic (Bondi) flow is considered a classic example of an accretion flow. This flow, however, is along a separatrix, which is usually not physically realizable. We demonstrate, using a pedagogical example, that it is the dynamics which selects the transonic flow.

DOI: 10.1103/PhysRevE.66.066303 PACS number(s): 47.40.Hg, 98.62.Mw, 05.45.−a

In astrophysics, the importance of accretion processes can hardly be overstated, especially in the context of the study of compact astrophysical objects and active galactic nuclei. Of such processes, a very important paradigm is that of the steady spherically symmetric flow, in which the motion of the accreting matter is steady and spherically symmetric, obeying the boundary condition that the bulk velocity fall off to zero at infinity, while the density asymptotically approaches a fixed value. Since this is the simplest situation in the class of accretion flows, it is the starting point in all relevant texts [1,2]. Studied extensively, among others, by Bondi [3], about 50 years ago, it also gives one of the clearest examples of a transonic flow (with which, in spherical symmetry, Bondi's name is associated), i.e., a flow in which the velocity is subsonic far away from the star and becomes supersonic as the surface of the star is approached. It is this simplicity and easy comprehensibility of the model that overshadows the fact that it is not encountered most often in practice. Accordingly, the spherically symmetric flow has been of continuing interest [4–7]. We first argue that the transonic (Bondi) solution of the steady spherically symmetric flow would not be physically realizable. This proposition is in contradiction to the position of Bondi himself, that ''the case physically most likely to occur is that with the maximum rate of accretion'' [3], which, in spherically symmetric accretion, is a case that is readily identified as the transonic solution [4]. We then argue that it is the dynamics which actually selects the transonic flow from among all possible trajectories.

The typical solutions for the spherically symmetric flow in the velocity-coordinate space is shown in Fig. 1 (to be read without the arrows). The two dark solid lines labeled **A** and **W** refer to the accretion flow, and the wind flow, respectively. The intersection point is an equilibrium point. The problem with Fig. 1 is that when it is read without the arrows it is slightly misleading. It does not show along what route an integration of dv/dr would proceed if we start with an initial condition $v=v_{in}$ at $r=r_{in}$ far away from the star. For a physically realizable flow, an initial condition infinitesimally close to a point on the accretion line **A** would trace out a curve infinitesimally close to **A** and in the limit would correctly reproduce **A**, evolving along it and passing through the equilibrium point (sonic point) as we integrate dv/dr, obtained from Euler's equation. We will show that the arrows on the integration route are as shown in Fig. 1. The direction of the arrows indicates that the spherically symmetric transonic accretion flow should not be physically realizable.

Is this a result confined to the spherically symmetric flow? We have verified, as we shall show briefly, that this holds also for the axisymmetric rotating accretion flow for a thin disc [8] which is a situation of practical interest. The technique for assigning the arrows remains identical to what we will show for the spherically symmetric case [9].

We wish to make our point with a fairly straightforward example. We consider the differential equation

$$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}=\frac{x+y-2}{y-x}. \tag{1}$$

An integral can be written down as

$$x^2-y^2-4x+2xy=-C, \tag{2}$$

where C is a constant. If we want that particular solution which passes through the point where $f(x,y)=g(x,y)=0$, namely $x=y=1$, then $C=2$. The curve $x^2-y^2-4x+2xy=-2$ factorizes into a pair of straight lines : $y-x(1+\sqrt{2})+\sqrt{2}=0$ and $y-x(1-\sqrt{2})-\sqrt{2}=0$. This pair is shown in Fig. 2 (to be read without the arrows) as the lines marked $\mathbf{A}'$ and $\mathbf{W}'$.

We want to explore the process of drawing the curve $\mathbf{A}'$ from a given initial condition. On the curve $\mathbf{A}'$, $y=0$ at $x=2+\sqrt{2}$. Let us begin with the initial condition $y=0$ at $x=2+\sqrt{2}-\epsilon$, where $0<\epsilon\ll 1$. The given initial condition fixes the constant C as $C=-2[1+\sqrt{2}\epsilon-\epsilon^2/2]$. Using this value of C, we can plot the curve given by Eq. (2). For a

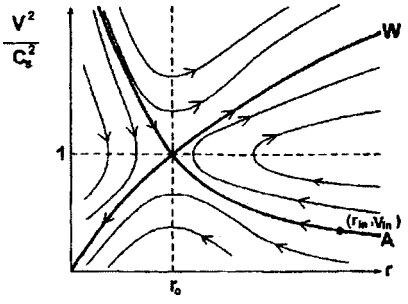

FIG. 1. Phase trajectories for spherically symmetric accretion onto a star. The bold solid lines, **A** and **W**, represent ''accretion'' and ''wind,'' respectively. The fixed point is at $r=r_0$, $v^2/c_s^2=1$. Linear stability analysis indicates that the fixed point of the flow is a saddle point. The direction of the arrows along the line **A** demonstrates that the transonic flow is not physically realizable.

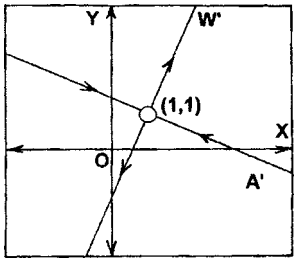

FIG. 2. Integration of Eq. (1) gives a pair of straight lines, with the integration constant fixed by the intersection point (1,1). In the figure the lines are marked $\mathbf{A}'$ and $\mathbf{W}'$. Parametrizing Eq. (1) as Eq. (5), we get the concept of arrows. Linear stability analysis indicates that for $\mathbf{A}'$ and $\mathbf{W}'$, the intersection point (1,1) is actually a saddle point, for which the arrows are as shown above.

given x, y is given by the relevant root of the quadratic equation thus obtained. The two roots are

$$y = x \pm \sqrt{2}[(x-1)^2 + \sqrt{2}\epsilon - \epsilon^2/2]^{1/2}. \tag{3}$$

Clearly, to satisfy $y=0$ at $x=2+\sqrt{2}-\epsilon$, the negative sign has to be chosen in Eq. (3), and we have

$$y = x - \sqrt{2}[(x-1)^2 + \sqrt{2}\epsilon - \epsilon^2/2]^{1/2}. \tag{4}$$

At $x=0$, $y=-\sqrt{2}(1+\sqrt{2}\epsilon-\epsilon^2/2)^{1/2}$, very different from $y=\sqrt{2}$, which one gets on the curve $\mathbf{A}'$. In the limit of $\epsilon \to 0$, one generates a part of $\mathbf{A}'$ ($x \geqslant 1$) and a part of $\mathbf{W}'$ ($x \leqslant 1$), instead of the curve $\mathbf{A}'$. Another way of stating this is the sensitivity to initial conditions in the drawing of $\mathbf{A}'$. If we make an error of an infinitesimal amount ϵ in prescribing the initial condition on $\mathbf{A}'$, i.e., we prescribe $y=0$ at $x=2+\sqrt{2}-\epsilon$ instead of $y=0$ at $x=2+\sqrt{2}$, then the "error" made at $x=0$ relative to $\mathbf{A}'$ is $2\sqrt{2}$ which is $O(1)$. An infinitesimal separation at one point leads to a finite separation at a point a short distance away. This is what we mean by saying that the curve $\mathbf{A}'$ (and similarly $\mathbf{W}'$) should not be physically realized.

The clearest and most direct understanding of the difficulty is achieved by writing Eq. (1) as the two differential equations

$$\frac{dy}{d\tau} = x + y - 2,$$

$$\frac{dx}{d\tau} = y - x, \tag{5}$$

where τ is some convenient parametrization. The fixed point of this dynamical system is (1,1), namely the point where $f(x,y)$ and $g(x,y)$ are simultaneously zero—the point through which $\mathbf{A}'$ and $\mathbf{W}'$ pass. Linear stability analysis of this fixed point shows that it is a saddle. The trajectories in this $y-x$ space can now be drawn with arrows and the result is as shown in Fig. 2. The distribution of the arrows (peculiar to a saddle) [10] implies $\mathbf{A}'$ and $\mathbf{W}'$ cannot be physically realized.

For the spherically symmetric flow, the relevant variables are the radial velocity v and the local density ρ. Ignoring viscosity, we write down Euler's equation for v as

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial V(r)}{\partial r}, \tag{6}$$

where P is the local pressure and V is the potential due to a gravitating body of mass M, i.e., $V=-GM/r$. The pressure is related to the local density through the equation of state $P=K\rho^{\gamma}$ where K is a constant, and $1<\gamma<\frac{5}{3}$, γ being the ratio of the specific heats at constant pressure and constant volume. The local density evolves according to the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v) = 0. \tag{7}$$

The stationary solution implies $\partial v/\partial t = \partial \rho/\partial t = 0$. The local sound speed is given by $c_s^2 = \partial P/\partial \rho = \gamma K \rho^{\gamma-1}$ and in the stationary situation, we can use Eq. (7) to write Eq. (6) as

$$\frac{dv^2}{dr} = \frac{2v^2}{r}\frac{\left(2c_s^2 - \frac{GM}{r}\right)}{v^2 - c_s^2}. \tag{8}$$

Integration of the equation with different initial conditions is supposed to generate the curves in Fig. 1. To assign arrows we write Eq. (8) in the parametrized form

$$\frac{dv^2}{d\tau} = 2v^2\left(2c_s^2 - \frac{GM}{r}\right),$$

$$\frac{dr}{d\tau} = r(v^2 - c_s^2). \tag{9}$$

The fixed point is at $r=r_0$, $v=v_0$ such that $v_0^2 = c_{s0}^2$ and $2r_0 c_{s0}^2 = GM$. It is immediately clear that the fixed point is the so called sonic point. We now carry out a linear stability analysis around the fixed point by writing $v^2 = v_0^2(1+\delta_1)$ and $r = r_0(1+\delta_2)$ and linearizing in δ_1 and δ_2. Straightforward algebra yields

$$\frac{d\delta_1}{d\tau} = 2c_{s0}^2[-(\gamma-1)\delta_1 + (6-4\gamma)\delta_2],$$

$$\frac{d\delta_2}{d\tau} = c_{s0}^2\left(\frac{\gamma+1}{2}\delta_1 + 2(\gamma-1)\delta_2\right). \tag{10}$$

The solutions for $\delta_{1,2}$ are of the form $e^{\lambda\tau}$, where λ is to be found from the roots of

$$\mathrm{Det}\begin{pmatrix} \lambda + 2(\gamma-1)c_{s0}^2 & 2(4\gamma-6)c_{s0}^2 \\ -\frac{\gamma+1}{2}c_{s0}^2 & \lambda - 2(\gamma-1)c_{s0}^2 \end{pmatrix} = 0. \tag{11}$$

The eigenvalues are found to be $\lambda = \pm c_{s0}^2\sqrt{2(5-3\gamma)}$. For $\gamma<\frac{5}{3}$, the fixed point is a saddle. Since the physical situa-

tions are restricted to this range, we find that the fixed point is always a saddle and hence the distribution of arrows will be as in Fig. 1.

For the more realistic axisymmetric flow, using R as the radial distance in the plane, the static equations of flow for an angular velocity Ω are (the equation of continuity is $\rho RHv = \text{const}$, where H is the disk thickness):

momentum balance,

$$\frac{dv^2}{dR} = 2\Omega^2 R - 2\Omega_K^2 - \frac{2}{\rho}\frac{dP}{dR}; \tag{12}$$

angular momentum balance,

$$v\frac{d}{dR}(\Omega R^2) = \frac{1}{\rho RH}\frac{d}{dR}\left(\frac{\alpha\rho c_s^2 R^3 H}{\Omega_K}\frac{d\Omega}{dR}\right), \tag{13}$$

where $\Omega_K^2 = GM/R^3$ and α is the effective viscosity of Shakura and Sunyaev [8]. An integral of motion follows from Eq. (13) and we can obtain an equation for dv^2/dR akin to Eq. (8), which we can write as the dynamical system

$$\frac{dv^2}{d\tau} = 2v^2\left(\frac{5c_s^2}{1+\gamma} - (\Omega_K^2 - \Omega^2)R^2\right),$$

$$\frac{dR}{d\tau} = R\left(v^2 - \frac{2c_s^2}{1+\gamma}\right). \tag{14}$$

An analysis identical to the one following Eq. (9) leads to the eigenvalues λ of the stability matrix which can easily be shown to be appropriate for a saddle [9].

We now focus on the fact that the real physical problem is dynamic as exhibited in Eqs. (6) and (7) and what we have tacitly assumed is that we can directly look at the stationary solution $v(r)$. As it turns out there are an infinite number of stationary solutions and if the static limit is taken directly, it is the Bondi solution which is very sensitive to the process of integration which determines the stationary trajectory. Several studies of the linear stability analysis of the stationary flows that have been carried out in the past have not really helped clarify the situation as it was finally established that among the stationary solutions, the transonic Bondi solution as well as all the subsonic solutions are stable in the linear stability sense [4,11]. Any selection mechanism then must be of a nonperturbative nature.

Accordingly, we return to our pedagogic example of Eq. (1) but now consider y as a field $y(x,t)$ with the evolution

$$\frac{\partial y}{\partial t} + (y-x)\frac{\partial y}{\partial x} = y + x - 2. \tag{15}$$

The stationary solution $y_0(x)$ satisfies Eq. (1) and the discussion that follows Eq. (1) is valid for $y_0(x)$. The stationary solutions $y_0(x)$ are as shown in Fig. (2) and the separatrices are $y_0(x) = x(1+\sqrt{2}) - \sqrt{2}$ and $y_0(x) = -x(\sqrt{2}-1) + \sqrt{2}$. It is easy to see that a linear stability analysis around the family of stationary solutions $y_0(x)$ shows an infinite number of them to be stable. We will now show that the dynamics actually preferentially selects the separatrices.

The general solution of Eq. (10) can be obtained by the method of characteristics [12]. The two independent characteristic curves of Eq. (10) are obtained from

$$\frac{dt}{1} = \frac{dx}{y-x} = \frac{dy}{y+x-2}. \tag{16}$$

They are

$$y^2 - 2xy - x^2 + 4x = C,$$

$$\left(x - 1 \pm \frac{1}{\sqrt{2}}(y-x)\right)e^{\mp\sqrt{2}t} = \tilde{C}. \tag{17}$$

The latter is obtained by integrating $dx/dt = y - x$ using the first integral.

With the choice of the upper sign, the solutions of Eq. (10) can be written as

$$y^2 - 2xy - x^2 + 4x = f\left[\left(x - 1 + \frac{1}{\sqrt{2}}(y-x)\right)e^{-\sqrt{2}t}\right], \tag{18}$$

where f is an arbitrary function. The function f is determined by the initial conditions. In this case we choose the initial condition that $y(x) = 0$ at $t = 0$ for all x. This leads to $f[x(1-1/\sqrt{2}) - 1] = -x^2 + 4x$ and yields

$$f(z) = Az^2 + Bz + C, \tag{19}$$

where

$$A = -\frac{2}{(\sqrt{2}-1)^2}, B = -\frac{4}{\sqrt{2}-1}, \quad C = 2. \tag{20}$$

With the initial condition $y(x) = 0$ at $t = 0$, the solution to Eq. (10) reads

$$[y - x(\sqrt{2}+1) + \sqrt{2}][y + x(\sqrt{2}-1) - \sqrt{2}]$$
$$= B\phi e^{-\sqrt{2}t} + A\phi^2 e^{-2\sqrt{2}t}, \tag{21}$$

where $\phi \equiv \phi(x,y) = x(1-1/\sqrt{2}) - 1 + y/\sqrt{2}$. Clearly as $t \to \infty$, the right hand side tends to zero and we approach one of the two separatrices shown in Fig. 2. Of the two possible separatrices, the one which will be relevant will be determined by some other requirement. For the astrophysical flow the two separatrices are the Bondi flow and the wind solution. One chooses the proper sign of the velocity to get the flow which one is interested in.

The mechanism for the selection of the asymptotes in Fig. 1 as the favored trajectories is identical. This can be appreciated from a look at Eq. (6). If the pressure term is ignored, we have a set of stationary solutions $v^2/2 - GM/r = C$. Which of these would be selected by the dynamics? If we start from $v = 0$ at $t = 0$ for all r, an identical reasoning to the one given above shows that it is the path with $C = 0$ which is selected.

The selection of $C=0$ for this case is dynamic in origin. The asymptotic solution of the differential equation $\dot{v}+v(\partial v/\partial r)=-GM/r^2$ by the method of characteristics yields

$$\frac{v^2}{2}-\frac{GM}{r}=-\frac{GM}{r-vt} \tag{22}$$

which for $t\to\infty$ goes to the static solution $v^2/2=GM/r$. This is also the choice according to the energy criterion. Corresponding to the initial condition given, the lowest possible total energy is $E=v^2/2-GM/r=0$ and the dynamics selects this particular stationary trajectory. The dynamic solution is nonperturbative in character and provides the mathematical justification for Bondi's assertion that the energy criterion should select the stationary flow. With the pressure term included, this argument leads to the selection of the transonic path (gravity always wins for small r). We note in passing the interesting fact that the temporal approach to the stationary path is power-law-like. This brings in interesting issues like long time tails which are characteristic of various nonequlibrium phenomena in statistical physics.

This research has made use of NASA's Astrophysics Data System. A conversation with S. Sengupta is also gratefully acknowledged. One of the authors (A.K.R.) would also like to acknowledge the financial assistance given to him by the Council of Scientific and Industrial Research, Government of India, and the help of his friends A. Bhattacharyay, Dr. T. K. Das, and D. Sanyal.

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